محاضرة رياضة ٥

Forbenius method

$$y'' + P(x)\, y' + Q(x)\, y = f(x)$$

$x_0 = 0$

o.p (ordinary point)

$P(0) \neq \infty$

$Q(0) \neq \infty$

$y = \sum a_n x^n$

ونقارن أقل أس واللى بعده والعام

(S.P) singular points

$P(0) = \infty$ or $Q(0) = \infty$

IRRegular singular point (I.s.p)

$x\,P(x)\big|_{x=0} = \infty$

or

$x^2\, Q(x)\big|_{x=0} = \infty$

ليس لها حل بالمسلسلات

Regular singular point (R.s.p)

$x\,P(x)\big|_{x=0} \neq \infty$

$\boxed{\text{and}}$

$x^2\, Q(x)\big|_{x=0} \neq \infty$

* نفرض الحل :-

$$y = \sum a_n x^{n+\lambda}$$

نعوض في المعادلة ونقارن معامل أقل أس ينتج معادلة في λ.

$$a\lambda^2 + b\lambda + c = 0$$

شكل الجذور

[1]

جذور غير متساوية

$$\lambda_1 \neq \lambda_2$$

$(\lambda_1 - \lambda_2)$ is ~~integer~~ non integer

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = \sum a_n x^{n+\lambda_2}$$

$$y_{G.s} = c_1 y_1 + c_2 y_2$$

[2]

$$\lambda_1 = \lambda_2 = \lambda$$

$$y_1 = \sum a_1 x^{n+\lambda}$$

ـ نحسب a_n في كل خطوة وبداخلها (λ) ثم نفاضل الناتج ونعوض في الصورة

$$y_2 = \left.\frac{\partial y(x,\lambda)}{\partial \lambda}\right|_{\lambda=\lambda_1}$$

$$y_{G.s} = c_1 y_1 + c_2 y_2$$

$$y_2 = y_1(x)\ln x + \sum \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda=\lambda_1} x^{n+\lambda}$$

Notes

$$y(x,\lambda) = \sum a_n(\lambda)\, x^{n+\lambda}$$

$$\left.\frac{\partial y}{\partial \lambda}\right|_{\lambda=\lambda_1} = \sum a_n x^{n+\lambda} (1) \ln x \Big|_{\lambda=\lambda_1} + \sum \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda=\lambda_1} x^{n+\lambda}$$

$\boxed{2}$ Lec 20

$\boxed{3}$ $\lambda_1 - \lambda_2 = \text{integer}$ الفرق عدد صحيح

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = \frac{\partial}{\partial \lambda}\left[(\lambda-\lambda_2)\, y(x,\lambda)\right]\Big|_{\lambda=\lambda_2}$$

$$y_{G.S} = C_1 y_1 + C_2 y_2$$

ـ نحسب y وبداخلها λ ونضرب الناتج في $(\lambda-\lambda_2)$ ونفاضل

$\boxed{Ex}$ solve : $4xy'' + 2y' + y = 0$ near $x_0 = 0$

$\boxed{Sol}$

$$y'' + P(x)\, y' + Q(x) = F(x)$$

$$y'' + \frac{1}{2x} y' + \frac{1}{4x} y = 0$$

$$P(x) = \frac{1}{2x} \quad ; \quad Q(x) = \frac{1}{4x}$$

$$P(0) = \infty \qquad Q(0) = \infty \implies (S.P)$$

$$x\,P(x)\Big|_{x=0} = x\left(\frac{1}{2x}\right)\Big|_{x=0} = \frac{1}{2} \neq \infty$$

$\boxed{3}$ Lec 2.

$$x^2 Q(x)\Big|_{x=0} = x^2\left(\frac{1}{4x}\right)\Big|_{x=0} = 0 \neq \infty$$

$\Rightarrow$ is R.S.P.

$$y = \sum a_n x^{n+\lambda}$$

$$y' = \sum (n+\lambda) a_n x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$$

$$4\sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-1} + 2\sum (n+\lambda) a_n x^{n+\lambda-1}$$
$$+ \sum a_n x^{n+\lambda} = 0$$

← لمعرفة أقل أس نضع $(n=0)$ في كل الحدود ونقارن معامل أصغر أس

← نضع $(n=0)$ في جميع المتسلسلات ونوجد أقل أس ومعامل

يعطى المعادلة المميزة في λ.

Co-eff of $x^{\lambda-1}$

$$4\lambda(\lambda-1)a_0 + 2\lambda a_0 + \text{لا يوجد} = 0$$

$$\lambda a_0 (2\lambda - 2 + 1) = 0$$

$a_0 \neq 0$

$\boxed{4}$ Lec 2.

$$\lambda_1 = 0 \quad \text{or} \quad \lambda_2 = \frac{1}{2}$$

← جذور غير متساوية والفرق يساوي عدد غير صحيح.

$$y_1 = \sum a_n x^{n+\lambda_1} \quad ; \quad y_2 = \sum a_n x^{n+\lambda_2}$$

Coeff. x^{λ}

$$4(\lambda+1)\lambda a_1 + 2(\lambda+1) a_1 + a_0 = 0$$

$$2(\lambda+1) a_1 \left[2\lambda + 1\right] = -a_0$$

$$a_1 = \frac{-a_0}{2(\lambda+1)(\lambda+1)} \longrightarrow (1)$$

Coeff. of $x^{n+\lambda}$

$$4(n+\lambda+1)(n+\lambda) a_{n+1} + 2(n+\lambda+1) a_{n+1} + a_n = 0$$

$$2(n+\lambda+1) a_{n+1} \left[2n + 2\lambda + 1\right] = -a_n$$

$$a_{n+1} = \frac{-a_n}{2(n+\lambda+1)(2n+2\lambda+1)}$$

• Put $n=1$

$$a_2 = \frac{-a_1}{2(\lambda+2)(2\lambda+3)} = \frac{a_0}{2(\lambda+2)(2\lambda+3)\,2(\lambda+1) * (2\lambda+1)}$$

نعوض في القوانين

Case(1) $\lambda_1 = 0$

$$y_1 = \sum a_n x^{n+\lambda_1} = \sum a_n x^n x^{\lambda_1}$$

$$= x^{\lambda_1}\left(a_0 + a_1 x + a_2 x^2 \cdots\cdots\right)\Big|_{\lambda=0}$$

$$y_1 = a_0 - \frac{a_0}{2}x + \frac{a_0}{(2)(3)(2)(2)}x^2 \cdots\cdots$$

$$y_1 = a_0\left[1 - \frac{x}{2} + \frac{x^2}{24} \cdots\cdots\right]$$

Case (2) $\lambda_2 = \frac{1}{2}$

$$y_2 = \sum a_n x^{n+\lambda_2} = \sum a_n x^n * x^{\lambda_2}$$

$$= x^{\frac{1}{2}}\left[a_0 + a_1 x + a_2 x^2 \cdots\cdots\right]$$

$$= x^{\frac{1}{2}}\left[a_\circ - \frac{a_\circ}{(2)\left(\frac{3}{2}\right)(2)}x + \frac{a_\circ}{2\left(\frac{5}{2}\right)(4)\left(\frac{3}{2}\right)(2)}x^2 \cdots\right]$$

$$y_{G.S} = C_1 y_1 + C_2 y_2$$

$\boxed{Ex}$ solve by series fn_s.

$$x^2 y'' - x y' + (x^2+1) y = 0 \qquad at \ x_\circ = 0$$

$\boxed{Sol}$

$$y'' + P(x) y' + Q(x) \cdot y = f(x)$$

$$y'' - \frac{1}{x} y' + \frac{x^2+1}{x^2} y = 0$$

$$P(x) = \frac{-1}{x} \quad ; \quad Q(x) = \frac{x^2+1}{x^2}$$

$$P(0) = \infty \quad ; \quad Q(0) = \infty \qquad is \quad S.P$$

$$x P(x)\Big|_{x=0} = x\left(\frac{-1}{x}\right)\Big|_{x=0} = -1 \neq \infty$$

$$x^2 Q(x)\Big|_{x=0} = x^2\left(\frac{-1}{x}\right)\Big|_{x=0} \neq \infty$$

$\Rightarrow$ R.S.P

$\boxed{7}$ Lec 20

$$y = \sum a_n x^{n+\lambda}$$

$$y' = \sum a_n (n+\lambda) x^{n+\lambda-1}$$

$$y'' = \sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda-2}$$

$$\sum (n+\lambda)(n+\lambda-1) a_n x^{n+\lambda} - \sum a_n (n+\lambda) x^{n+\lambda}$$

$$+ \sum a_n x^{n+\lambda+2} + \sum a_n x^{n+\lambda} = 0$$

أقل أس من x^{λ}

* Coeff. of x^{λ}

$\cancel{\lambda(\lambda-1)a_n - \lambda a_n + a}$

$\lambda(\lambda-1)a_0 - \lambda a_0 +$ لايوجد $+ a_0 = 0$

$$a_0 \left[\lambda^2 - 2\lambda + 1\right] = 0$$

$a_0 \neq 0$

$$(\lambda-1)(\lambda-1) = 0$$

$\lambda_1 = 1 \quad , \quad \lambda_2 = 1$ جذور متساوية

$$y_1 = \sum a_n x^{n+\lambda_1}$$

$$y_2 = y_1 \ln x + \sum \left.\frac{\partial a_n}{\partial \lambda}\right|_{\lambda = \lambda_1} x^{n+\lambda_1}$$

$*$ Coeff of $x^{\lambda+1}$

$(\lambda+1)\lambda a_1 - (\lambda+1) a_1 +$ لا يوجد $+ a_1 = 0$

$a_1[(\lambda+1)(\lambda-1) + 1] = 0 \Rightarrow a_1 = 0$

$\lambda_1, \lambda_2 = 1$ معناها إنه ما بداخل القوس لا يساوي صفر.

$*$ Coeff. of $x^{n+\lambda}$

$$(n+\lambda)(n+\lambda-1)a_n - (n+\lambda)a_n + a_{n-2} + a_n = 0$$

$$a_n\Big[(n+\lambda)\underbrace{(n+\lambda-1-1)}_{((n+\lambda)-2)} + 1\Big] = -a_{n-2}$$

$$a_n = \frac{-a_{n-2}}{(n+\lambda)^2 - 2(n+\lambda)+1}$$

$$a_n = \frac{-\,a_{n-2}}{(n+\lambda-1)^2}$$

$n = 2$

$$a_2 = -\frac{a_0}{(\lambda+1)^2}$$

$n = 3$

$$a_3 = \frac{-\,a_1}{(\lambda+2)^2} = 0$$

Case (1) $\quad \lambda_1 = 1 \;;\; y_1 = \sum a_n x^{n+\lambda_1}$

$$y_1 = x^{\lambda_1} \sum a_n x^n \Big|_{\lambda=\lambda_1} = x\left[a_0 + a_1 x + a_2 x^2 \cdots\cdots\right]\Big|_{\lambda=1}$$

$$= x\left[a_0 + 0 - \frac{a_0}{4} x^2 \cdots\cdots\right]$$

Case (2) $\quad \lambda_2 = 1 \;;\; y_2 = y_1 \ln x + \sum \frac{\partial a_n}{\partial \lambda} x^{n+\lambda_1} \Big|_{\lambda=\lambda_2}$

$$y_2 = y_1 \ln x + x^{\lambda_2}\left[\frac{\partial a_0}{\partial \lambda} + \frac{\partial a_1}{\partial \lambda} x + \frac{\partial a_2}{\partial \lambda} x^2 \cdots\cdots\right]$$

$$a_1 = 0 \Rightarrow \frac{\partial a_1}{\partial \lambda} = 0$$

$$a_2 = \frac{-a_0}{(\lambda+1)^2} = -a_0 (\lambda+1)^{-2}$$

← تعامل مع a_0 كأنه ثابت
ولا توضع تفاضلها بالنسبة لـ λ
تساوي صفر.

$$\frac{\partial a_2}{\partial \lambda} = +2(\lambda+1)^{-3} a_0$$

← ممكن نوضحها $a_0^* \Leftarrow$

$$a_0^* = \frac{\partial a_0}{\partial \lambda}$$

$$a_2 = 2(\lambda+1)^{-3} a_0 - (\lambda+1)^{-2} a_0^* \Big|_{\lambda=1}$$

$$a_2 = \frac{2a_0}{8} - \frac{a_0^*}{4}$$

بالتعويض في y_2

$$y_2 = y_1 \ln x + x\left[a_0^* + 0 + \left(\frac{2a_0}{8} - \frac{a_0^*}{4}\right)x^2 + \cdots\right]$$

$$y_{G.S} = c_1 y_1 + c_2 y_2$$

Homework

Solve ~~the~~ by series fn

$$x^2 y'' + x y' + (x^2 - 1) y = 0$$

near $x_0 = 0$

$\boxed{12}$ Lec 20